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The quasi–abelian limit

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Abstract. A new, non perturbative, eikonal method called the "quasi abelian limit" (QAL) is suggested for high energy quark (nucleon) scattering involving the exchange of all possible, non interacting, non abelian gluons (mesons). With this method, those functional integrals defining, e.g., the exchange of color cooordinates in quark–quark scattering, are replaced by a finite number of quadratures over a subset of their coordinates. Mathematically, this procedure is not rigourous, because an unjustified interchange of limits has been performed; physically, it corresponds to the observation that the non perturbative sum over all color–moment fluctuations can vanish at arbitrarily high energies. The QAL generates a result in agreement with a corrected, "contiguity" calculation, when the latter is summed over all perturbative orders.

1 Introduction

Ordered exponentials (OEs) appear in Green's functions, and in the scattering amplitudes constructed from them, in a natural and fundamental way, for both abelian and non abelian interactions [1]. In the former situation, the exchange of (soft) quanta, with 4–momenta considerably less than those of the incident particles, removes the need for ordering; while in the latter case the same kinematical ("eikonal") regimes will still require ordering because of the presence of isotopic (in $SU(2)$) or color (in $SU(3)$) degrees of freedom. Each scattering particle appears with its own OE, and the amplitude constructed from the exchange of an arbitrary number of virtual mesons or gluons requires the linkage of both OEs, an entity which can be described as a "doubly–ordered" exponential. Because of the complexity of such forms, a perturbative approach has been the most common method of proceeding from Lagrangian to scattering amplitude [2].

Recently, a functional approach called "contiguity" was devised [3] to extract the leading–log(E) (LL) terms (of the special subset of graphs corresponding to the exchange of non–interacting mesons between scattering fermions) in every order of perturbation theory for the eikonal function, rather than for the amplitude; and this method has also been applied in a special model which deals with gluon–string exchange between a pair of scattering quarks [4]. Because of the special form of the effective propagator found in [4], it became apparent that there was another, and different way to perform not only the contiguity identification of the relevant LL term, but also to sum over all perturbative orders of those terms, in a direct and non perturbative calculation of the amplitude in the limit of high energy; this was the origin of the method there called quasi–abelian.

It was since been realized – and it will be displayed in this paper – that a natural form of rescaled limit in the high energy region may be applicable to any such non abelian theory; and the second example to which it was applied – the isotopic, $SU(2)$ scattering amplitude of [3] – gave an answer different from that obtained by summing over all the perturbative, contiguity contributions. It was then discovered that an error had been made in the course of the summations, such that the stated eikonal of [3] does not, in fact, display "reggeized" energy and impact parameter behavior; and that the corrected, contiguity summations are compatible with the far simpler, QAL calculation of the amplitude. Identification of the error made in [3], the general definition of the QAL, and its use in calculating the amplitude, are the subjects of this paper.

Our starting point is the exact, functional expression for the scattering amplitude of a pair of quarks (each in a different hadron), interacting via the exchange of all possible (interacting) gluons and closed quark loops. The only initial approximation made is that one which defines the word " eikonal", in which the quark momenta (p, p') entering into the mass shell amputated quark Green's functions $G[A]$ are treated as much larger than any of the Fourier components k of the gluon fields $A(k)$. This and only this is what is meant by the words "eikonal approximation". It should be noted that we are treating the scattering quarks as if they were asymptotic particles, suppressing the fact that they are each to be considered as bound to a pair of spectator quarks.

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We use techniques invented decades ago [1] to insure time reversal invariance, calculating not the amplitude T, but rather its variation with respect to the quark–gluon couplings, g_1 and g_2 of quark #1, q_I and quark #2, q_{II} respectively:

$$T = \int_0^g dg_1 \int_0^g dg_2 \, \frac{\partial^2 T}{\partial g_1 \partial g_2}$$

and performing the integrations over g_1 and g_2 at a later stage.

At a convenient point in this process, we shall restrict all gluon exchanges to those originating on one quark and terminating on the other. Abelian experience has shown that small momentum transfer limits of graphs corresponding to "self linkage" along either line are not to be taken seriously, except to generate "physical" masses and couplings, which we shall call m and g, respectively, suppressing any variation of the latter with respect to momentum of its associated gluon; and we shall, without apology, make the same assumption here, for this process of small angle scattering, where $s >> |t|$. Gauge invariance for the final, relatively simple subset of graphs evaluated to illustrate the QAL, is discussed in Appendix B.

The amplitude so described is presumably very close to the goal of many modern attempts now underway [5]. We believe that the present effort, starting from first (functional) principles of quantum field theory, is worth making, and for several reasons. Firstly, it is appropriate to develop any non perturbative approximations which simplify the analysis, such as the QAL of the present paper. Many older calculations [6] have discussed in perturbative fashion estimates of relevant non–abelian amplitudes which appear in this problem; we have the same objective, but wish to pursue the analysis in a non perturbative, functional manner, developing the needed techniques as we go along.

For the final, relatively simple subset of graphs that we do estimate in this paper, our analysis differs from that of the first two sets of papers quoted in ref [6], because we are not interested in the specifically IR limit as the gluon mass vanishes, and also because we are here neglecting "self–linkages" (generating t dependence which damps the amplitude) included by these authors in their search for IR singularities. A perturbative development of the graphs that we do calculate brings in well–defined, leading log factors of $\ln(E/m)$[2], which can be summed for values of $x \simeq g \ln(E/m)$ of the order of unity; but instead of exponentiating, one finds a result of qualitative form $(1+x)^{-1}$, which "self cancels" for large x and removes the E dependence so carefully calculated in each perturbative order. By adopting the QAL for these graphs, one calculates from the outset a non perturbative set of quadratures whose perturbative components are just the original set of graphs without the $\ln(E/m)$ dependence that has self canceled. In fact, this may be thought of as a partial justification of an effective, $1/N$ expension, in which certain, specific, non abelian structure is neglected.

In fact, such "self cancellation" can be seen in the explicit calculations of the Carruthers and Zachariasen paper of ref [6], whose amplitudes vanish as E/m becomes very large. This is necessary and appropriate for the analysis of IR singularities undertaken by these authors; but suggests that in a non IR situation, the leading log terms self cancel, requiring an investigation of the next to leading terms, and so forth. The QAL, in contrast and from the beginning, drops these terms as irrelevant for the truly important, leading log structure of the non IR amplitude, which comes from very different graphs corresponding to towers and their generalizations.

Secondly, even for abelian estimates, there are honest differences of opinion as to the possible importance of those contributions to the eikonal function which appear when multi–t–channel, ("vertical") gluons are exchanged between the scattering quarks, with each such gluon exchanging all possible numbers of ("horizontal") gluons between them. This is a most non trivial generalization of the original "towers" of Cheng and Wu, and of Chang and Yan, and many others [2], and was extensively discussed two decades ago [7,8]. It is an old problem, which has not been simplified by non abelian complications, and is one that, it seems clear to us, can only be resolved by a non perturbative, functional analysis. The present paper, beginning with the "exact" eikonal amplitude immediately below, and using relatively simple graphs to define the QAL, sets the stage for other work (presently underway) towards this goal.

We begin, then, with the statement of the exact QCD generating functional $\mathcal{Z}\{j, \eta, \bar{\eta}\}$ as a functional of gluon and quark sources, $j^a_\mu(z)$, $\eta^b_\alpha(x)$, $\bar{\eta}^c_\beta(y)$, respectively, and in an axial gauge specified by the constant 4–vector n_μ:

$$\mathcal{Z}\{j, \eta, \bar{\eta}\} = N \int d[A] \, \delta[n.A] \, e^{-\frac{i}{4}\int F^2} \, e^{L[A]} \, e^{i\int \bar{\eta} G_c[A] \eta} \quad (1.1)$$

where N is a normalization constant so chosen that $\mathcal{Z}\{0,0,0\} = 1$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$, $G_c[A]$ defines the (potential theory) Green's function of a quark propagating in a background gluonic field $A^a_\mu(x)$ and $L[A]$ represents the closed quark loop functional, $L[A] = -Tr \ln \Big[G_c[A] G_c^{-1}[0] \Big]$.

Passing from generating functional to S–matrix element in the standard way [8], but extended to quarks in the $s >> |t|$ limit, the mass shell amputated qq scattering amplitude becomes:

$$\begin{aligned}
\frac{\partial^2 T}{\partial g_1 \partial g_2} &= -i \frac{p_1 \cdot p_2}{m^2} \int d^4 z_1 \, e^{i q_1 \cdot z_1} \int d^4 z_2 \, e^{i q_2 \cdot z_2} \, N \\
&\times \int d[A] \, \delta[n \cdot A] e^{-\frac{i}{4}\int F^2 + L[A]} \\
&\times \left(e^{i g_1 \int_0^{+\infty} ds \; p'_{1\mu} \cdot A^{a'}_\mu (z_1 - s p'_1) \lambda^I_{a'}} \right)_+ \\
&\times \lambda^I_a \left(e^{i g_1 \int_{-\infty}^{0} ds \; p_{1\mu} \cdot A^{a'}_\mu (z_1 - s p_1) \lambda^I_{a'}} \right)_+ \\
&\times \left(e^{i g_2 \int_0^{+\infty} ds \; p'_{2\nu} \cdot A^{b}_\nu (z_2 - s p'_2) \lambda^{II}_{b}} \right)_+ \\
&\times \lambda^{II}_a \left(e^{i g_2 \int_{-\infty}^{0} ds \; p_{2\nu} \cdot A^{b}_\nu (z_2 - s p_2) \lambda^{II}_{b}} \right)_+
\end{aligned} \quad (1.2)$$

where the λ_a^I, λ_b^{II} are the Gell–Mann color matrices, and p and p' are the 4–momenta of incident and final quarks, respectively; here $q_1 = p_1 - p_1'$, $q_2 = p_2 - p_2'$ and the instantaneous position coordinates of q_I and q_{II} are denoted by z_1 and z_2.

What must now be done is to convert the functional operations of (1.2) into a form that can be explicitly evaluated.This we shall do by invoking the QAL, but in two steps, the first of which exhibits the physical content of the QAL by starting in Sect. 2 from the conventional eikonal representation of a scattering amplitude given in terms of an impact parameter sum over the eikonal function; while the second step, in Appendix B, describes the QAL passage from (1.2) to the conventional eikonal representation, (2.1) and (2.2). We emphasize, here as well as subsequently in the text, that the great simplifications resulting from the QAL do involve a mathematically non rigourous interchange of limits. We suggest, however, that the terms neglected by this interchange of limits, correspond to those which display self–cancellation at very high energies; and as such, the QAL becomes a sensible and convenient approximation method, with application far beyond the simple graphs employed in this paper.

2 Formulation

Corresponding to the sum of all possible virtual mesons/gluons exchanged between a pair of fermions/quarks, we write the generic form of an eikonal representation of the (large energy/small momentum transfer) scattering amplitude as:

$$T(s,t) = \frac{is}{2m^2} \int d^2b\ e^{\,iq.b} \left[1 - e^{\,i\chi(s,b)}\right] \qquad (2.1)$$

with:

$$e^{\,i\chi} = \quad e^{\mathcal{D}_{I,II}} \left(e^{\,ig\int_{-\infty}^{+\infty} ds_1\ p_1\cdot A_I^a(z_1 - s_1 p_1)\lambda_a^I} \right)_+ \times \left(e^{\,ig\int_{-\infty}^{+\infty} ds_2\ p_2\cdot A_{II}^b(z_2 - s_2 p_2)\lambda_b^{II}} \right)_+ \Bigg|_{A_{I,II}=0} \qquad (2.2)$$

where $z_{1,2}$ and $p_{1,2}$ are the incident configuration space and 4–momentum coordinates of the interacting fermions number 1 and 2, respectively ; here the linkage operator is given by:

$$\mathcal{D}_{I,II} = -i \int \frac{\delta}{\delta A_{I\mu}^a(u)}\, D_{c,\,\mu\nu}^{ab}(u-w)\, \frac{\delta}{\delta A_{II\nu}^b(w)}$$

and the propagator of the exchanged meson is denoted by $D_{c,\,\mu\nu}^{ab}$. The conventional scattering invariants are given by $s = -(p_1 + p_2)^2 = 4E^2$, and $t = -(p_1 - p_1')^2 = -q^2$, m is (for simplicity) the same mass of fermion 1 and 2, $b = (z_1 - z_2)_t$ is the impact parameter, with the subscript t denoting the transverse (perpendicular to the incoming fermion momenta) separation of the two fermions, each of energy E, in their CM.

It should be emphasized that an integration over coupling constants, here suppressed, must really be performed before one can identify the RHS of (2.2) as $e^{\,i\chi}$; but, for our purposes, (2.2) does express the qualitatively correct form of that functional operation which represents the linkage of a pair of fermions by the exchange of all numbers of virtual mesons, and we shall refer to the logarithm of the RHS of (2.2) as the "eikonal" of this problem. This question, as well as a derivation of (2.1), and a demonstration of gauge invariance, is discussed in Appendix B.

In order to perform the functional operation of (2.2), each OE may be rewritten as a functional integration over variables $\alpha_a(s_1)$, $u_a(s_1)$ and $\beta_b(s_2)$, $v_b(s_2)$, whose effect is to isolate the A–dependence of each OE in an ordinary exponential; for example:

$$N' \int d\,[\alpha]\ \int d\,[u]\ e^{\,i\int ds\, \alpha_a(s) u_a(s)} \left(e^{\,i\int ds\, \lambda_a^I\, u_a(s)} \right)_+ \times e^{\,-ig\int ds\ p_1\cdot A_I^a(z_1 - sp_1)\,\alpha_a(s)} \qquad (2.3)$$

where all $\int ds_i$ integrations run over the same, infinite interval as that of the original OE. The normalization N' is defined such that the functional integral:

$$N' \int d\,[\alpha]\ e^{\,i\int ds\, \alpha_a(s)\, \left[u_a(s) - g\, p_1\cdot A_I^a(z_1 - sp_1)\right]}$$

is the delta functional $\delta\left[u_a(s) - g\, p_1 \cdot A_I^a(z_1 - sp_1)\right]$, and the subsequent $\int d\,[u]$ reproduces exactly the OE of (2.2). If the interval $\int ds$ is broken up into very small intervals labelled by s_k of width Δs, where n factors of Δs equal the size of the integration range, then $N' = \left(N_k'\right)^n$, where

$$N_k' = \left(\Delta s / 2\pi\right)^D \qquad (2.4)$$

and D is the number of dimensions over which each $\int d^D\,[u(s_k)]$ runs.

Of course, this separation does not solve the non abelian problem, but only postpones the evaluation of the OE of (2.3) until a later stage. The A dependence, however, is now effectively abelian, and the linkage operation of (2.2) upon it can be performed exactly; and one easily finds:

$$e^{\,i\chi} = N' \int d\,[\alpha] \int d\,[u]\ e^{\,i\int ds\, \alpha_a(s) u_a(s)} \left(e^{\,i\int ds\, \lambda_a^I\, u_a(s)} \right)_+ \times N' \int d\,[\beta] \int d\,[v]\ e^{\,i\int ds\, \beta_b(s) v_b(s)} \left(e^{\,i\int ds\, \lambda_b^{II}\, v_b(s)} \right)_+ \times e^{\,ig^2 \int\int_{-\infty}^{+\infty} ds_1\, ds_2\, p_{1\,\mu}\, \alpha_a(s_1)\, D_{c,\,\mu\nu}^{ab}(z - s_1 p_1 + s_2 p_2)\, p_{2\,\nu}\, \beta_b(s_2)} \qquad (2.5)$$

where $D_{c,\,\mu\nu}^{ab}(z - s_1 p_1 + s_2 p_2)$ is the meson propagator, and $z = z_1 - z_2$. Were this an abelian problem, the α_a, β_b factors mulptiplying D_c would be replaced by unity, as would be the remaining functional integrals of (2.5), and the result would be the well known, abelian eikonal:

$$i\chi = -i \frac{g^2}{2\pi}\, \gamma(s)\, K_0(\mu\, b) \qquad (2.6)$$

where $\gamma(s) = (s - 2m^2)(s(s - 4m^2))^{-1/2}$ is that factor depending on the spin of the exchanged boson, of mass μ.

The idea and mechanism of contiguity, as defined and illustrated in [3], is not at question here; rather, the error made in that paper occurred when the "nested" LL contributions to the perturbative eikonal of order $g^{2n}/n!$ were multiplied by another term corresponding to the permutation of all color indices of that order; and from the latter term a factor of $n!$ was omitted. The summation over all n quoted was therefore of form $\exp(x)$, with $x = K\ln(E/m)$, $K = (g^2/2\pi)\,K_0(\mu b)$, rather than the geometric sum of form $[1+x]^{-1}$, defined for $|x| < 1$, a sum which must then be continued into the region of $|x| > 1$, and which does not then display any net, LL behavior, in the limit of arbitrarily high energy.

In effect, the LL(E) dependence "self cancels", and for this situation, it is difficult to have any confidence in the idea of summing leading logs. Do the next–to–leading logs also cancel ? The LL terms in perturbation theory that lead to "exponentiation", or "reggeization", are those associated with "towers" of closed quark loops and/or gluon–gluon interactions, interactions which have both been suppressed here. By the QAL analysis to follow, we shall find that the absence of energy dependence for this class of graphs is realized at an early stage, immediately upon performing that limiting process which defines the QAL method.

There are several methods of approach to the present problem, each of which leads to the idea that the QAL is intuitively reasonable at ultra–high energies. Consider first the last line of the eikonal expression (2.5), which in the limit of large energy and small momentum transfer contains the CM 4–momenta $p_1 = (0, 0, +E; iE)$ and $p_2 = (0, 0, -E; iE)$; and introduce the following rescaled, proper–time variables $\bar{s}_1 = E\,s_1$, $\bar{s}_2 = E\,s_2$. This exponential factor then becomes:

$$ig^2 \int\!\!\int_{-\infty}^{+\infty} d\bar{s}_1\, d\bar{s}_2\, \frac{p_{1\mu}}{E}\, \alpha_a(\frac{\bar{s}_1}{E})\, D^{ab}_{c,\,\mu\nu} \times\Big(z - \bar{s}_1(\frac{p_1}{E}) + \bar{s}_2(\frac{p_2}{E})\Big)\frac{p_{2\,\nu}}{E}\,\beta_b(\frac{\bar{s}_2}{E}) \qquad (2.7)$$

The ratios $p_{1,2}/E$ are independant of E, and the only visible, overt energy dependence of (2.7) is that of the arguments of α_a and β_b. Imagine that a calculation – e.g. contiguity – is now carried out using (2.7), and that at the end of that calculation, the limit $E \to \infty$ is taken. For the case of gluon–string exchange (where the D_c above is replaced by the Q of [4]), it has been shown that the LL terms one finds correspond precisely to the limit that follows from taking $E \to \infty$ under the $s_{1,2}$ integrals of the equation which corresponds to (2.7). Here, that limit would correspond to the replacement of (2.7) by:

$$ig^2\,\alpha_a(0)\,\beta_b(0)\,\frac{p_{1\mu}}{E}\,\frac{p_{2\,\nu}}{E} \times \int\!\!\int_{-\infty}^{+\infty} d\bar{s}_1\, d\bar{s}_2\, D^{ab}_{c,\,\mu\nu}\Big(z - \bar{s}_1(\frac{p_1}{E}) + \bar{s}_2(\frac{p_2}{E})\Big)$$

or, by what is the same (re–rescaled) thing:

$$ig^2\,\alpha_a(0)\,\beta_b(0)\,p_{1\mu}\,p_{2\,\nu} \times \int\!\!\int_{-\infty}^{+\infty} ds_1\, ds_2\, D^{ab}_{c,\,\mu\nu}\Big(z - s_1\,p_1 + s_2\,p_2\Big) \qquad (2.8)$$

in the limit of extremely large E.

This is one definition of the quasi–abelian limit, where one imagines that the non perturbative result is correctly described by (2.8); several other approaches, which have the same QAL consequence, are gathered together in Appendix A. Mathematically, until one learns how to estimate corrections to this limit, the procedure is surely an unjustified interchange of limiting operations, for one is supposed to calculate all functional integrals before allowing E to become arbitrarily large. Physically, this interchange suggests that sums and averages over all parameters of color exchange will, at very high energies, occur in the same way, and need be calculated just once – at $s_{1,2} = 0$ – because not enough proper time is available for fluctuations in the possible methods of color transfer; one might say [9] that the sum of all "color moments" effectively vanishes as $E \to \infty$. That is, regardless of the space–time point along a quark's trajectory where a virtual gluon is emitted or absorbed, the variables describing that color exchange are those associated with the quarks' distance of closest approach.

3 Application

Immediately, one sees from (2.8) that all the leading $\ln\big(E/m\big)$ dependence of this eikonal must cancel – as is the case when the $SU(2)$ contiguity forms are properly calculated and summed – because the $s_{1,2}$ integrals of (2.8) are just those leading to (2.6), which is independent of CM energy in the limit $\big(E/m\big) \to \infty$. But of far greater significance is the observation that, if the limit is correct, the only values of $s_{1,2}$ which can enter into non trivial functional integrals over $\alpha_a, \beta_b, u_a, v_b$, are those of $s_{1,2} = 0$. Breaking up these integrals into discrete integrations over averaged variables carrying the values s_{1i}, and s_{2j}, integration over all the other $s_{1i} \neq 0 \neq s_{2j}$ intervals gives, after extracting the proper parts of the normalization factors N', precisely a factor of unity. Each OE is replaced by an un–ordered exponential factor depending on either $u_a(0)$ or $v_b(0)$, e.g.:

$$\left(e^{\,i\int_{-\infty}^{+\infty} ds\,\lambda_a\,u_a(s)}\right)_+ = e^{\,i\Delta s\,\lambda_{a_n}\,u_{a_n}(s)} \cdots e^{\,i\Delta s\,\lambda_{a_1}\,u_{a_1}(s)} \cdot e^{\,i\Delta s\,\lambda_a\,u_a(0)} \cdots e^{\,i\Delta s\,\lambda_{a_{-n}}\,u_{a_{-n}}(s)}\Big|_{\substack{s_n > \cdots > s_1 > s_0 > s_{-1} > \cdots > s_{-n} \\ s_n \to +\infty,\, s_{-n} \to -\infty}}$$

which gives:

$$\left(e^{\,i\int_{-\infty}^{+\infty} ds\,\lambda_a\,u_a(s)}\right)_+ = e^{\,i\Delta s\,\lambda_a\,u_a(0)}$$

because the integrals $\int d^D\alpha(s_{1i})$ and $\int d^D\beta(s_{2j})$ for $s_{1i} \neq 0 \neq s_{2j}$ produce factors of $\delta\big(u_a(s_{1i})\big)$ and $\delta\big(v_b(s_{2j})\big)$, so that each OE reduces to the un–ordered form above.

For $SU(2)$, for simplicity, with the D of (2.4) chosen as 3, what remains is the set of quadratures:

$$e^{i\chi} = \left(\frac{\Delta s}{2\pi}\right)^6 \int d^3\alpha(0) \int d^3\beta(0) \int d^3u(0) \times \int d^3v(0)\, e^{i\Delta s\left[\alpha_a(0)u_a(0)+\beta_b(0)v_b(0)\right]} \times e^{\frac{i}{2}\sigma_a^I \Delta s\, u_a(0)} \cdot e^{\frac{i}{2}\sigma_b^{II}\Delta s\, v_b(0)} \cdot e^{-i K\alpha_a(0)\beta_a(0)} \tag{3.1}$$

where $K = (g^2/2\pi)K_0(\mu b)$, and we have chosen $D^{ab}_{c,\,\mu\nu} = \delta_{ab}\delta_{\mu\nu}\Delta_c$. A trivial change of variables: $\Delta s\, u_a(0) = u_a$, $\Delta s\, v_b(0) = v_b$, $\alpha_a(0) = \alpha_a$, $\beta_b(0) = \beta_b$, converts (3.1) into:

$$e^{i\chi} = (2\pi)^{-6} \int d^3\alpha \int d^3\beta \int d^3u \int d^3v\, e^{i(\alpha\cdot\mathbf{u}+\beta\cdot\mathbf{v})-i\alpha\cdot\beta K} \times e^{\frac{i}{2}\sigma^I\cdot\mathbf{u}} \cdot e^{\frac{i}{2}\sigma^{II}\cdot\mathbf{v}} \tag{3.2}$$

Integration over $\int d^3\alpha \int d^3\beta$ is easily performed, yielding, after another rescaling:

$$e^{i\chi} = (2\pi)^{-3} \int d^3u \int d^3v\, e^{i(\mathbf{u}\cdot\mathbf{v})} \cdot e^{\frac{i}{2}\sigma^I\cdot\mathbf{u}\sqrt{K}} \cdot e^{\frac{i}{2}\sigma^{II}\cdot\mathbf{v}\sqrt{K}} \tag{3.3}$$

Were the $\sigma_{I,II}$ of (3.3) treated as ordinary numbers, the remaining integrals would immediately generate what might be called the naive result:

$$e^{i\chi_0} = e^{-i(\sigma_I\cdot\sigma_{II})K/4}\,; \tag{3.4}$$

however, a more careful, if elementary, evaluation of (3.4) is needed, which yields:

$$e^{i\chi_0} = \cos\left(\frac{K}{4}\right) - \left(\frac{K}{4}\right)\sin\left(\frac{K}{4}\right) - \frac{i}{3}(\sigma_I\cdot\sigma_{II})\left[\sin\left(\frac{K}{4}\right) + \left(\frac{K}{4}\right)\cos\left(\frac{K}{4}\right)\right] \tag{3.5}$$

Since the product $\sigma_I \cdot \sigma_{II}$ has eigenvalues of $+1$ (triplet state) and -3 (singlet state), the singlet eikonal function is then given by:

$$e^{i\chi_S} = \left(1 + i\frac{K}{4}\right) e^{iK/4} = \rho\, e^{i\left(\frac{K}{4}-\theta\right)} \tag{3.6}$$

where $\rho = \left[1 + (K/4)^2\right]^{1/2}$ and $\theta = \tan^{-1}(K/4)$. Using the same notation, the triplet eikonal may be written as:

$$e^{i\chi_T} = \frac{\rho}{3}\, e^{i\left(\frac{K}{4}-\theta\right)} + \frac{2\rho}{3}\, e^{-i\left(\frac{K}{4}-\theta\right)} \tag{3.7}$$

Assuming the validity of the QAL, equations (3.6) and (3.7), not the results following from (3.4), nor those quoted in [3], are the correct expressions for these eikonals.

Generalizations from $SU(2)$ to $SU(3)$ are possible although somewhat tedious; here, for any $SU(N)$, a convenient representation for needed exponentials can be written in the form:

$$e^{i\lambda\cdot u} = \frac{1}{N}\sum_n \left[1 + \lambda_a \frac{\partial r_n}{\partial u_a}\right] e^{ir_n} \tag{3.8}$$

where the r_n are the eigenvalues of the matrices $\lambda\cdot\mathbf{u}$. For $SU(3)$, for example, one must solve the triple equations:

$$\sum_n r_n = 0, \quad \sum_n r_n^2 = a\,(\delta_{ab}\,u_a\,u_b)$$
$$\sum_n r_n^3 = b\,(d_{abc}\,u_a\,u_b\,u_c)$$

where a and b are real constants, for the three r_n, which is equivalent to finding the roots of a relevant cubic equation; for $SU(2)$, one has, immediately : $e^{\frac{i}{2}\sigma\cdot\mathbf{u}} = \cos\left(\frac{u}{2}\right) + i\,\frac{\sigma\cdot\mathbf{u}}{u}\sin\left(\frac{u}{2}\right)$.

When the method is extended to closed quark loop and gluon–gluon interactions, where an imaginary part increasing with E develops in the quantity analogous to the K of this computation, these quadratures will require – as in [4] – an analytic continuation, which must be performed in such a manner that unitarity is respected. Initial application of the QAL method to closed quark loops suggests that the simplifications noted above will also hold for any loops containing gluons that couple directly (or indirectly, through the medium of other closed loops) to the scattering quarks. It may also be mentioned that a generalization of the present QAL technique can be formulated for wide angle, quark scattering at high energies, which retains appropriate spin–dependence needed to generate the observed spin–correlation ratios which have been obtained in a growing number of experiments [10], and which still demand a consistent theoretical explanation. Work is presently underway on both of these applications of the QAL.

If, by a careful estimate of corrections to the unjustified interchange which produces the QAL, one were to find that the QAL generates a reasonable approximation even at finite energies, the method would then become applicable to the calculation, or estimation, of all QCD correlation functions, for a wide variety of processes.

4 Summary

As noted in Appendix B, when one explicitly constructs the functions that display color dependence, e.g. $\alpha_a(s)$, one always finds that dependence entering into an integral of form $p_{1\mu}\int_{-\infty}^{+\infty} ds\, \alpha_a(s)\, f_\mu^a(sp_1)$. The simplest QAL argument is then, that upon rescaling, this is identical to $p_{1\mu}/E \int_{-\infty}^{+\infty} d\bar{s}\, \alpha_a(\bar{s}/E)\, f_\mu^a(\bar{s}p_1/E)$, and in the limit $E/m \to \infty$ can be rewritten as $(p_{1\mu}/E)\,\alpha_a(0) \times \int_{-\infty}^{+\infty} d\bar{s}\, f_\mu^a(\bar{s}p_1/E)$, assuming that $\alpha_a(s)$ may be treated as a continuous function of s. Of course, this is mathematically incorrect; but if the very large $\ln(E/m)$ terms which are neglected by the QAL do self–cancel, then the QAL may be used to bypass a huge amount of useless calculation.

The corrected contiguity calculation, without the QAL assumption, provides one example of just how this can work. Here, all of the α, β generated leading log dependence sums up to a form which, when continued to arbitrarily high $\ln(E/m)$, removes itself (in the sense of $\lim(1+x)^{-1} \to 0$ as $x \to \infty$) from the problem. The simple and succinct way of bypassing all the effort of that example is to adopt the QAL. Physically, it is as if all color fluctuations and exchanges of color coordinates between scattering quarks take place only in the very short period about their point of closest approach, or equivalently, at the proper time corresponding to that point of closest approach.

We expect the QAL to have extensive applications to the more interesting, gluon–interaction and closed quark loop eikonal graphs, which are now under active consideration.

Appendix A

We here briefly describe a variety of other approaches to the QAL, which all appear to become equivalent in the limit of ultra–high energy. It should first be noted that the functional integrals over the α_i and β_j variables may be performed exactly, with a result that (2.5) may be put into the (asymmetric but convenient) form:

$$\prod_{i,j}(2\pi)^{-D}\int d^D u_i \int d^D v_j\, e^{-iu_i v_j} \cdots e^{i\lambda^I\cdot\sum_k \Delta_{ik} u_k} \cdots e^{i\lambda^{II}\cdot v_j} \cdots \tag{A.1}$$

where the dots in (A.1) indicates the exponential λ^I, λ^{II} factors with all the differing values of s_{1i} and s_{2j}. In particular, the sum over the t_k indices of each u_k exponential coefficient, for a particular value of s_{1i}, may be written in the continuum limit as:

$$\begin{aligned} &i\ \Delta s(p_1\cdot p_2)\int_{-\infty}^{+\infty} dt\, \Delta_c(z - s_{1i}\,p_1 + tp_2)u_a(t) \\ &= i\Delta s\,(p_1\cdot p_2)\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{ik_\perp\cdot b} \\ &\times \int \frac{dk_3}{(2\pi)}\, e^{ik_3 z_3}\int \frac{dk_0}{(2\pi)}\, e^{-ik_0 z_0}\, \frac{e^{-iEs_{1i}(k_3-k_0)}}{\mu^2+k_\perp^2+k_3^2-k_0^2} \\ &\times \int_{-\infty}^{+\infty} dt\, e^{-iEt(k_3+k_0)} u_a(t) \end{aligned} \tag{A.2}$$

Rescaling the last t–integrand of (A.2), by $\bar{t} = E\,t$, one considers:

$$\frac{1}{E}\int_{-\infty}^{+\infty} d\bar{t}\, e^{-i\bar{t}\,(k_3+k_0)}\, u_a(\bar{t}/E) \tag{A.3}$$

which, under the naive interchange of limits introduced here, becomes $(2\pi/E)\delta\,(k_3+k_0)\,u_a(0)$, as $E \to \infty$. Were the $u_a(t)$ a smooth, analytic function of its argument, one could argue that the first correction to this result would involve an extra factor of E^{-1}, and could therefore be dropped. But, although written in continuous form, the $u_a(t)$ represents a function which is at best piecewise continuous, and no statement can be made about its derivatives. Yet, one has the intuitive feeling that – could the entire functional integral be performed properly – the limit of large E should involve only the variable $u_a(0)$; and it is this intuition, aided by the relative simplicity of the result, which makes the QAL so attractive. Combining these last statements, the only, non trivial contribution to the functional integral comes from those coordinates corresponding to zero proper–time arguments; and the result may easily be transformed into that of the text, (3.3).

There is a second, independent method of arriving at the QAL, which begins by the observation that the perturbative LL terms, as provided by the arguments of [2] and [3], arise as the coefficients of multiple commutators:

$$\left[\lambda_{a_1},\ \left[\lambda_{a_2},\cdots\left[\lambda_{a_{n-1}},\ \lambda_{a_n}\right]\cdots\right]\right]$$

describing the well–known fact that, e.g., the product of two such exponential factors is not equal to the exponential of their sum:

$$e^{i\lambda\cdot a}\, e^{i\lambda\cdot b} \neq e^{i\lambda\cdot(a+b)}$$

But, if we believe the corrected contiguity result, in which the LLE dependence vanishes as $E \to \infty$, let us neglect all such commutator dependence, and replace in (A.1): $\prod_i e^{i\lambda^I\cdot\sum_k \Delta_{ik}u_k}$ by $e^{i\lambda^I\cdot\sum_{i,k}\Delta_{ik}u_k}$ or, in the continuum limit, by:

$$e^{ig^2\,(p_1\cdot p_2)\int\int_{-\infty}^{+\infty} ds_1\, ds_2\, \Delta_c(z-s_1p_1+s_2p_2)\, u(s_2)\cdot\lambda^I}$$

Integration over ds now generates a $\delta\,(k_3-k_0)$, which permits $\int dk_0$ to be performed, and generates the simple result $\exp\left(-K\lambda_a^I u_a(t_0)\right)$, with $t_0 = (z_3 - z_0)/2E$. It is not even necessary to assume the $E \to \infty$ limit here, for the only non trivial contribution to (A.1) can come when s_{1i} and s_{2j} are both equal to t_0, and that contribution is exactly the same as that of (3.3).

A third method of approach to the QAL utilises the fall off (or rapidly oscillating) behavior of the propagator $\Delta_c(z)$, which assuming spacelike dependence, for simplicity, decreases roughly as $\exp\left(-\mu(Z^2)^{1/2}\right)$ for large values of $Z^2 = (z - s_1p_1 + s_2p_2)^2$. In our case, for $s_{1,2} \neq 0$, this means a propagator fall off roughly as $\exp\left(-\mu E\, s_{1i}s_{2j}\right)$, where we have neglected other, less important terms (which become important if one of the s variables is zero) decreasing as $\exp\left(-\mu(E\,s\,z_{3,0})^{1/2}\right)$. In terms of the original s_{1i}, s_{2j} variables, the region of importance is within the "star" formed by the hyperbolas $|s_{1i}\,s_{2j}| < (\mu E)^{-2}$; all other s_{1i}, s_{2j} will give a negligible contribution to the product $\alpha(s_{1i})\,\Delta_{ij}\,\beta(s_{2j})$.

It should be noted that arbitrarily large α_i, β_j values are allowed, but then the entire interaction exponential oscillates to zero. But the point of the exercise is that, in the limit $E \to \infty$, the "star" shrinks to the point $s_{1i} \sim s_{2j} \sim 0$, so that, again, only these are the significant values. The same type of analysis can be carried out starting from (A.1), and leads to the same QAL result, as above.

Of course, as emphasized in the text, the really crucial question of how to estimate corrections to the QAL remains open; and it is only after this step has been performed, that one may feel confident about this interchange of limits.

Appendix B

To the best of our knowledge, a derivation of the exact, eikonal amplitude, proceeding from the variation of couplings $g_{1,2}$ in the non Abelian case has not yet been presented; and that shall be the aim of this Appendix. Only at the very end will we specialize to the simplest sub–set of graphs used to define the QAL.

We begin, then, with an exact form of the 4–point, configuration space, quark–quark scattering amplitude, obtained from the exact generating function of (1.1):

$$\begin{aligned} M(x_1,y_1;x_2,y_2) &= i\,N\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]} \\ &\cdot G_c^I(x_1,y_1|g_1A)\cdot G_c^{II}(x_2,y_2|g_2A) \end{aligned} \tag{B.1}$$

displaying the unrenormalized quark–gluon couplings $g_{1,2}$; here $F_{\mu\nu}$ and $L[A]$ depend on g, which value is subsequently approached by $g_{1,2}$. We next calculate:

$$\begin{aligned} \frac{\partial^2 M}{\partial g_1 \partial g_2} &= -i\,N\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]} \\ &\cdot\int d^4z_1\,G_c^I(x_1,z_1|g_1A)\,\gamma_\mu^I\,A_\mu^a(z_1)\,\lambda_a^I\,G_c^I(z_1,y_1|g_1A) \\ &\cdot\int d^4z_2\,G_c^{II}(x_2,z_2|g_2A)\,\gamma_\nu^{II}\,A_\nu^b(z_2)\,\lambda_b^{II}\,G_c^{II}(z_2,y_2|g_2A) \end{aligned} \tag{B.2}$$

and follow this by mass shell amputation on all four quark legs (but only on one leg for each, no recoil Green's function !), generating for the Fourier transform of (B.2) the quantity:

$$\begin{aligned} \frac{\partial^2 T}{\partial g_1 \partial g_2} &= i\,N\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]} \\ &\cdot\int d^4z_1\,e^{iq_1\cdot z_1}\left(e^{ig_1\int_0^{+\infty}ds\,p_1'\cdot A(z_1-sp_1')\lambda^I}\right)_+ \\ &\cdot\left(\frac{p_1}{m}\cdot A(z_1)\lambda^I\right)\left(e^{ig_1\int_{-\infty}^0 ds\,p_1\cdot A(z_1-sp_1)\lambda^I}\right)_+ \\ &\cdot\int d^4z_2\,e^{iq_2\cdot z_2}\left(e^{ig_2\int_0^{+\infty}ds\,p_2'\cdot A(z_2-sp_2')\lambda^{II}}\right)_+ \\ &\cdot\left(\frac{p_2}{m}\cdot A(z_2)\lambda^{II}\right)\left(e^{ig_2\int_{-\infty}^0 ds\,p_2\cdot A(z_2-sp_2)\lambda^{II}}\right)_+ \end{aligned} \tag{B.3}$$

where we have replaced the (B.2) factors γ_μ^I, γ_ν^{II} by $-i\frac{p_{1\mu}}{m}$, $-i\frac{p_{2\nu}}{m}$ in the small momentum transfer limit.

The exponential factors of (B.3) are those familiar from abelian, eikonal models, except that here, because of the non commuting λ matrices, ordered exponentials (OEs) must be used. Each quark line OE of (B.3) may be rewritten as in the text, so as to separate its A dependence from the color matrices; for example:

$$\begin{aligned} &\left(e^{ig_1\int_0^{+\infty}ds\,p_1'\cdot A(z_1-sp_1')\lambda^I}\right)_+\lambda_a^I \\ &\times\left(e^{ig_1\int_{-\infty}^0 ds\,p_1\cdot A(z_1-sp_1)\lambda^I}\right)_+ \\ &= N'\int d[\alpha]\int d[u]\,e^{i\int_{-\infty}^{\infty}ds\,\alpha_a(s)u_a(s)}\left(e^{i\int_0^\infty ds\,\lambda^I\cdot u(s)}\right)_+ \\ &\times\lambda_a^I\left(e^{i\int_{-\infty}^0 ds\,\lambda^I\cdot u(s)}\right)_+ \\ &\cdot e^{-ig_1\int_{-\infty}^{+\infty}ds\,\alpha_{a'}(s)\left[\theta(s)\,p_{1\mu}'A_\mu^{a'}(z_1-sp_1')+\theta(-s)\,p_{1\mu}A_\mu^{a'}(z_1-sp_1)\right]} \end{aligned} \tag{B.4}$$

where N' is a normalization constant chosen so that functional integration over the $\alpha_{a'}(s)$ produces exactly the delta functional:

$$\begin{aligned} \delta\Big[u_{a'}(s)-g_1\Big(\theta(s)\,p_{1\mu}'A_\mu^{a'}(z_1-sp_1') \\ +\theta(-s)\,p_{1\mu}A_\mu^{a'}(z_1-sp_1)\Big)\Big] \end{aligned}$$

and the subsequent $\int d\,[u]$ reproduces exactly the LHS of (B.1). We shall adopt a similar representation for the other quark's OE factors, with $\beta(s)$, $v(s)$ replacing $\alpha(s)$ and $u(s)$.

Another way of writing the A dependence of RHS of (B.4) is:

$$\exp\Big[-ig_1\int d^4z\,J_{1\mu}^a(z)\,A_\mu^a(z)\Big]$$

where $J_{1\mu}^a(z)$ represents the color current of that quark:

$$\begin{aligned} J_{1\mu}^a(z) = \int_{-\infty}^{+\infty}ds\,\alpha_a(s)\Big[p_{1\mu}'\delta^{(4)}(z-z_1+sp_1')\,\theta(s) \\ +p_{1\mu}\delta^{(4)}(z-z_1+sp_1)\,\theta(-s)\Big] \end{aligned} \tag{B.5}$$

and in the customary, small t, eikonal amplitude, we neglect in (B.5) the momentum transfer q_1, so that:

$$J_{1\mu}^a(z) = p_{1\mu}\int_{-\infty}^{+\infty}ds\,\alpha_a(s)\,\delta^{(4)}(z-z_1+sp_1) \tag{B.6}$$

With these replacements, (B.3) becomes:

$$\begin{aligned} \frac{\partial^2 T}{\partial g_1 \partial g_2} &= \frac{i\,N}{m^2}\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]} \\ &\cdot\int d^4z_1\,e^{iq_1\cdot z_1}\int d^4z_2\,e^{iq_2\cdot z_2} \\ &\cdot N'^2\int d[\alpha]\int d[u]\int d[\beta]\int d[v]\;e^{i\int[\alpha\cdot u+\beta\cdot v]} \\ &\cdot\big(p_1\cdot A^a(z_1)\big)\big(p_2\cdot A^b(z_2)\big)\,e^{i\int[g_1J_1+g_2J_2]\cdot A} \\ &\cdot\left(e^{i\int_0^\infty ds\,\lambda^I\cdot u}\right)_+\lambda_a^I\left(e^{i\int_{-\infty}^0 ds\,\lambda^I\cdot u}\right)_+ \\ &\cdot\left(e^{i\int_0^\infty ds\,\lambda^{II}\cdot v}\right)_+\lambda_b^{II}\left(e^{i\int_{-\infty}^0 ds\,\lambda^{II}\cdot v}\right)_+ \end{aligned} \tag{B.7}$$

The A factors of (B.7) may be written more compactly as:

$$\left(\frac{1}{ig_1}\, p_{1\mu} \frac{\delta}{\delta J^a_{1\mu}(z_1)}\right) \times \left(\frac{1}{ig_2}\, p_{2\nu} \frac{\delta}{\delta J^b_{2\nu}(z_2)}\right) e^{\,i\int[g_1 J_1 + g_2 J_2]\cdot A}$$

and the $\lambda^{I,II}$ dependence as:

$$\frac{1}{i}\frac{\delta}{\delta u_a(0)}\,\frac{1}{i}\frac{\delta}{\delta v_b(0)}\left(e^{\,i\int_{-\infty}^{+\infty}\lambda^I\cdot u}\right)_+ \cdot \left(e^{\,i\int_{-\infty}^{+\infty}\lambda^{II}\cdot v}\right)_+$$

Further, an integration by parts on the $u_a(0)$, $v_b(0)$ variables then converts the functional derivatives $\frac{1}{i}\frac{\delta}{\delta u_a(0)} \times \frac{1}{i}\frac{\delta}{\delta v_b(0)}$ into multiplicative factors of $\alpha_a(0)\beta_b(0)$, so that we obtain the compact form:

$$\frac{\partial^2 T}{\partial g_1 \partial g_2} = \int d^4 z_1\, e^{\,iq_1\cdot z_1} \cdot \int d^4 z_2\, e^{\,iq_2\cdot z_2}\left(\alpha_a(0)\,\frac{p_{1\mu}}{g_1}\,\frac{\delta}{\delta J^a_{1\mu}(z_1)}\right) \cdot \left(\beta_b(0)\,\frac{p_{2\nu}}{g_2}\,\frac{\delta}{\delta J^b_{2\nu}(z_2)}\right) R\Big[g_1 J_1, g_2 J_2\Big] \quad \text{(B.8)}$$

where:

$$R = -\frac{i\,N}{m^2}\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2 + L[A]} \cdot N'^2 \int d[\alpha]\int d[u]\int d[\beta]\int d[v]\; e^{\,i\int[\alpha\cdot u + \beta\cdot v]} \cdot \left(e^{\,i\int \lambda^I\cdot u}\right)_+ \left(e^{\,i\int\lambda^{II}\cdot v}\right)_+ e^{\,i\int[g_1 J_1 + g_2 J_2]\cdot A} \quad \text{(B.9)}$$

It is now convenient to make use of the simple and obvious functional relations:

$$g_1\,\frac{\partial R}{\partial g_1} = \int d^4 z\; J^a_{1\mu}(z)\,\frac{\delta}{\delta J^a_{1\mu}(z)} R \quad \text{(B.10)}$$

and:

$$g_2\,\frac{\partial R}{\partial g_2} = \int d^4 w\; J^b_{2\nu}(w)\,\frac{\delta}{\delta J^b_{2\nu}(w)} R \quad \text{(B.11)}$$

Substituting the definition of $J_{1\mu}(z)$ into (B.10), the latter becomes:

$$g_1\,\frac{\partial R}{\partial g_1} = p_{1\mu}\int_{-\infty}^{+\infty} ds\,\alpha_a(s)\,\frac{\delta R}{\delta J^a_{1\mu}(z_1 - sp_1)} \quad \text{(B.12)}$$

and upon operation on both sides of (B.10) by $\int d^4 z_1\, e^{\,iq_1\cdot z_1}$, one obtains:

$$\int d^4 z_1\, e^{\,iq_1\cdot z_1} g_1\,\frac{\partial R}{\partial g_1} = \int_{-\infty}^{+\infty} ds\,\alpha_a(s)\int d^4 z_1\, e^{\,iq_1\cdot z_1}\, p_{1\mu}\,\frac{\delta R}{\delta J^a_{1\mu}(z_1 - sp_1)} \quad \text{(B.13)}$$

On the RHS of (B.13) make the variable change $z_1 - sp_1 = z_1'$, so that:

$$\int d^4 z_1\, e^{\,iq_1\cdot z_1}\,\frac{\partial R}{\partial g_1} = \frac{p_{1\mu}}{g_1}\int_{-\infty}^{+\infty} ds\,\alpha_a(s)\,e^{\,isq_1\cdot p_1} \cdot \int d^4 z_1'\, e^{\,iq_1\cdot z_1'}\,\frac{\delta R}{\delta J^a_{1\mu}(z_1')} \quad \text{(B.14)}$$

where we henceforth drop the prime of z_1'. A similar relation must hold for variations of g_2 and J_2; or, for both together:

$$\int d^4 z_1\, e^{\,iq_1\cdot z_1}\int d^4 z_2\, e^{\,iq_2\cdot z_2}\,\frac{\partial^2 R}{\partial g_1 \partial g_2} = \frac{p_{1\mu}}{g_1}\,\frac{p_{2\nu}}{g_2}\int_{-\infty}^{+\infty} ds_1\,\alpha_a(s_1)\,e^{\,is_1 q_1\cdot p_1} \cdot \int_{-\infty}^{+\infty} ds_2\,\beta_b(s_2)\,e^{\,is_2 q_2\cdot p_2} \cdot \int d^4 z_1\, e^{\,iq_1\cdot z_1}\int d^4 z_2\, e^{\,iq_2\cdot z_2}\,\frac{\delta^2 R}{\delta J^a_{1\mu}(z_1)\,\delta J^b_{2\nu}(z_2)} \quad \text{(B.15)}$$

The RHS of (B.15) is almost the same combination as that which appears on the RHS of (B.8), except for the factors:

$$\int\int_{-\infty}^{+\infty} ds_1\, ds_2\,\alpha_a(s_1)\,\beta_b(s_2)\,e^{\,i(s_1 q_1\cdot p_1 + s_2 q_2\cdot p_2)}$$

of (B.15), which differ from the $\alpha_a(0)\,\beta_b(0)$ of (B.8). Let us therefore multiply both sides of (B.8) by $(2\pi)^2\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$, and note that the resulting RHS:

$$\frac{p_{1\mu}}{g_1}\,\frac{p_{2\nu}}{g_2}\,\alpha_a(0)\,\beta_b(0)\int_{-\infty}^{+\infty} ds_1\, e^{\,is_1 q_1\cdot p_1}\int_{-\infty}^{+\infty} ds_2\, e^{\,is_2 q_2\cdot p_2} \int d^4 z_1\, e^{\,iq_1\cdot z_1}\int d^4 z_2\, e^{\,iq_2\cdot z_2}\,\frac{\delta^2 R}{\delta J^a_{1\mu}(z_1)\,\delta J^b_{2\nu}(z_2)} \quad \text{(B.16)}$$

closely resembles the RHS of (B.15) except that the latter's $\alpha_a(s_1)\,\beta_b(s_2)$ are replaced by $\alpha_a(0)\,\beta_b(0)$. One must also ask if it is permitted to multiply T by $\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$, to which a positive response will be demonstrated below. We now argue that in the limit of sufficiently large E/m, the $\alpha_a(s_1)\,\beta_b(s_2)$ factors of (B.15) may be replaced by $\alpha_a(0)\,\beta_b(0)$; this argument, which is at the heart of the QAL, is made in several different contexts in Appendix A, and we use its simplest form. In general, $\alpha_a(s)$ dependence is always displayed in the form:

$$p_{1\mu}\int_{-\infty}^{+\infty} ds\,\alpha_a(s)\,f^a_\mu(sp_1)$$

One now rescales the dummy s variable to $\bar{s} = Es$, so that this combination becomes:

$$\frac{p_{1\mu}}{E}\int_{-\infty}^{+\infty} d\bar{s}\,\alpha_a(\bar{s}/E)\,f^a_\mu(\bar{s}p_1/E)$$

where, in the CM, $p_{1\mu} = (0,0,E,iE)$ in the large E/m limit (and, similarly, $p_{2\nu} = (0,0,-E,iE)$). If we treat

$\alpha_a(s)$ as a continuous function of s, then in the limit $E/m \to \infty$, one would expect this combination to be equivalent to:

$$\frac{p_{1\mu}}{E}\,\alpha_a(0)\int_{-\infty}^{+\infty} d\bar{s}\, f^a_\mu(\bar{s}p_1/E)$$

or, upon a rescaling, to be $p_{1\mu}\,\alpha_a(0)\int_{-\infty}^{+\infty} ds\, f^a_\mu(sp_1)$. We argue in the text and in Appendix A that, physically, this limit corresponds to color transfers ocurring at very small proper times, which, by a judicious choice of CM coordinate system, corresponds to a small relative distance of the scattering quarks. This is the QAL, and its adoption here means that the RHS of (B.15) is equivalent to the RHS of (B.8) multiplied by $(2\pi)^2\,\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$. The latter combination must then be equivalent to the LHS of (B.15), i.e.:

$$(2\pi)^2\,\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)\;\frac{\partial^2 T}{\partial g_1\partial g_2} = \int d^4z_1\, e^{\,iq_1\cdot z_1}\int d^4z_2\, e^{\,iq_2\cdot z_2}\,\frac{\partial^2 R}{\partial g_1\partial g_2}$$

After integration over the couplings, $\int_0^g dg_1\int_0^g dg_2$, one has:

$$\begin{aligned}&(2\pi)^2\,\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)\;T\\ &\quad=\frac{i\,N}{m^2}\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]}\\ &\quad\cdot\int d^4z_1\,e^{\,iq_1\cdot z_1}\int d^4z_2\,e^{\,iq_2\cdot z_2}\\ &\quad\cdot N'^2\int d[\alpha]\int d[u]\int d[\beta]\int d[v]\;e^{\,i\int[\alpha\cdot u+\beta\cdot v]}\\ &\quad\cdot\left(e^{\,i\int\lambda^I\cdot u}\right)_+\left(e^{\,i\int\lambda^{II}\cdot v}\right)_+\left[1-e^{\,i\int g[J_1+J_2]\cdot A}\right]\end{aligned}\qquad\text{(B.17)}$$

where the additive constant of integration (the "1") inside the bracket of (B.17) insures that T vanishes if $g\to 0$.

The RHS of (B.17) may be rewritten as:

$$\frac{i}{m^2}\int d^4z_1\,e^{\,iq_1\cdot z_1}\int d^4z_2\,e^{\,iq_2\cdot z_2}\left[1-e^{\,i\chi}\right]\qquad\text{(B.18)}$$

where:

$$\begin{aligned}e^{\,i\chi}&=N\int d[A]\,\delta[n\cdot A]\,e^{-\frac{i}{4}\int F^2+L[A]}N'^2\\ &\cdot\int d[\alpha]\int d[u]\int d[\beta]\int d[v]\;e^{\,i\int[\alpha\cdot u+\beta\cdot v]}\\ &\cdot\left(e^{\,i\int\lambda^I\cdot u}\right)_+\left(e^{\,i\int\lambda^{II}\cdot v}\right)_+\;e^{\,ig\int[J_1+J_2]\cdot A}\end{aligned}\qquad\text{(B.19)}$$

and where the coefficient of the integration constant is exactly unity, as explained in the discussion of normalization following (B.32). Note that if the QAL had not been made, the result of (B.17) would not take the form of an eikonal representation at this early stage; rather, one would have to postpone integration over the couplings until after the gluon fluctuations $\int d[A]$ are performed; and the result need not, in general, take the expected, eikonal form.

We next turn to the appropriateness of the multiplication of T by $\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$; or, equivalently, to ask if T may be defined with kinematic values of its variables so that $q_1\cdot p_1 = q_2\cdot p_2 = 0$. Let us imagine a continuation of the quark masses $m_{1,2}$ and $m'_{1,2}$ inside T, where m_i and m'_i represent initial and final masses, continued so that $m_i\neq m'_i$. Specifically, for $\mathbf{q}_{1\perp}\neq 0$ in the CM, we define $q_{1,3}=(\mathbf{q}_{1\perp})^2/2E$ (the value this quantity takes after the mass continuation is removed), and require $q_{1,0}=({m_1}^2+{m_2}^2-{m'_1}^2-{m'_2}^2)/2E$ (which is a valid relation, independent of the mass values); and we then continue the masses so that ${m_1}^2+{m_2}^2-{m'_1}^2-{m'_2}^2=(\mathbf{q}_{1\perp})^2$. In this way, $q_{1,3}=q_{1,0}$, so that $q_1\cdot p_1=0$. Similar definitions may be made for $q_{2,3}=-(\mathbf{q}_{2\perp})^2/2E$, and $q_{2,0}=({m_1}^2+{m_2}^2-{m'_1}^2-{m'_2}^2)/2E$, assuming that $\mathbf{q}_1+\mathbf{q}_2=0$; and with this choice of parameters, $q_2\cdot p_2=0$. Here, $q_{1,0}+q_{2,0}\to 0$ only as the masses return to their normal values; but in the limit of very large E, $q_{1,0}$ and $q_{2,0}$ both vanish, as does their sum.

Overall 4 momentum conservation is achieved by translationnal invariance, which requires that $\chi(z_{1,2})=\chi(z)$, with $z=z_1-z_2$; and hence the $z_{1,2}$ integrations can be rewritten as:

$$\begin{aligned}&(2\pi)^4\,\delta^{(4)}(q_1+q_2)\int d^4z\,e^{\,iq_1\cdot z}\left[1-e^{\,i\chi(z)}\right]\\ &\quad=(2\pi)^4\,\delta^{(4)}(q_1+q_2)\int dz_0\int dz_3\,e^{\,iq_{1,3}\cdot z_3-iq_{1,0}\cdot z_0}\\ &\quad\cdot\int d^2b\,e^{\,i\mathbf{q}_{1,\perp}\cdot\mathbf{b}}\left[1-e^{\,i\chi(z)}\right]\end{aligned}\qquad\text{(B.20)}$$

Further, all the z dependence of χ enters in the combination $z-s_1p_1+s_2p_2$, where $s_{1,2}$ are integration variables which may be redefined so as to remove from this combination all explicit $z_{3,0}$ dependence (and whose new limits of integration can be extended to $\pm\infty$ as $E\to\infty$); and in this way, χ can depend only on $b=z_1-z_2$, so that (B.20) generates:

$$(2\pi)^4\,\delta^{(4)}(q_1+q_2)\,(2\pi)^2\,\delta(q_3)\,\delta(q_0)\int d^2b\,e^{\,iq_1\cdot b}\left[1-e^{\,i\chi(b)}\right]$$

The factor standing to the left of the $\int d^2b$ can be cast into the form:

$$(2\pi)^4\,\delta^{(4)}(q_1+q_2)\,(2E)^2\,\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$$

so that common factors of $(2\pi)^2\,\delta(q_1\cdot p_1)\,\delta(q_2\cdot p_2)$ may be cancelled from both sides of (B.17), or of (B.18), and the quark masses continued back to their proper values. Suppressing the explicit factor $(2\pi)^4\,\delta^{(4)}(q_1+q_2)$ of T, one finds from (B.18) the standard eikonal representation:

$$T=\frac{is}{2m^2}\int d^2b\,e^{\,i\mathbf{q}\cdot\mathbf{b}}\left[1-e^{\,i\chi}\right],\qquad s=4E^2\qquad\text{(B.21)}$$

Finally, it is appropriate to rewrite the gluon fluctuation $\int d[A]$ in a way which exhibits the effects resulting from our choice of axial gauge. For this, it is convenient to first write:

$$e^{-\frac{i}{4}\int F^2} = e^{-\frac{i}{4}\int f^2} \cdot e^{-\frac{i}{4}\int (F^2-f^2)} \tag{B.22}$$

and, firstly, to concentrate on the first RHS factor of (B.22). Here, $f^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, resembling QED field strengths, with a color index. Rewriting this factor in the form:

$$e^{-\frac{i}{2}\int A^a_\mu(-\partial^2)A^a_\mu} \cdot e^{+\frac{i}{2}\int(\partial_\mu A^a_\mu)^2}\ , \tag{B.23}$$

the second term of (B.23) may be replaced by:

$$N_\psi \int d[\psi]\, e^{-\frac{i}{2}\int \psi_a^2 + i\int A^a_\mu \partial_\mu \psi_a}\ , \tag{B.24}$$

where N_ψ is an integration constant given by $N_\psi^{-1} = \int d[\psi]\, e^{-\frac{i}{2}\int \psi_a^2}$. Substitution into (B.19) then produces:

$$\begin{aligned} e^{i\chi} = N'^2 &\int d[\alpha]\int d[u]\int d[\beta]\int d[v]\ e^{i\int[\alpha\cdot u+\beta\cdot v]} \\ &\cdot\left(e^{i\int \lambda^I\cdot u}\right)_+ \left(e^{i\int \lambda^{II}\cdot v}\right)_+ \\ &\cdot N\, N_\psi \int d[\psi]\, e^{-\frac{i}{2}\int\psi^2} N_\theta \int d[\theta] \\ &\cdot \int d[A]\, e^{-\frac{i}{2}\int A(-\partial^2)A}\ e^{i\int \mathcal{J}^a_\mu A^a_\mu}\ \mathcal{F}[A] \end{aligned} \tag{B.25}$$

where $\mathcal{J}^a_\mu = g\left(J^a_{1\mu} + J^a_{2\mu}\right) + \partial_\mu\psi_a + n_\mu\theta_a$, $\mathcal{F}[A] = e^{L[A]-\frac{i}{4}\int(F^2-f^2)}$ and where we have used the representation $\delta[n\cdot A] = N_\theta \int d[\theta]\, e^{i\int n_\mu \theta_a A^a_\mu}$ with N_θ an appropriate normalization constant. Finally, it will be convenient to introduce a functional Fourier representation $\tilde{\mathcal{F}}[\phi]$ of $\mathcal{F}[A]$:

$$\mathcal{F}[A] = \int d[\phi]\, \tilde{\mathcal{F}}[\phi]\, e^{i\int \phi^a_\mu A^a_\mu}$$

so as to group all the A dependence of (B.25) into the gaussian form:

$$\begin{aligned} &N\, N_\psi\, N_\theta \int d[A]\, e^{-\frac{i}{2}\int A(-\partial^2)A}\ e^{i\int(\phi+\mathcal{J})A} \\ &\quad = N\, N_A^{-1}\, N_\psi\, N_\theta\ \exp\Big[\frac{i}{2}\int(\phi+\mathcal{J})D_c\,(\phi+\mathcal{J})\Big] \end{aligned} \tag{B.26}$$

where $N_A^{-1} = \int d[A]\, e^{-\frac{i}{2}\int A(-\partial^2)A}$, and where $\tilde{D}^{ab}_{c,\mu\nu}(k) = \delta_{\mu\nu}\delta_{ab}/k^2$ represents the causal, momentum space inverse of $(-\partial^2)$.

In the exponential factor of (B.26), we isolate the explicit $(n\theta + \partial\psi)$ dependence, replacing the former by:

$$\begin{aligned} &\exp\Big[\frac{i}{2}\int(n\theta+\partial\psi)D_c\,(n\theta+\partial\psi)\Big] \\ &\cdot\exp\Big[i\int(\phi+J)D_c\,(n\theta+\partial\psi)\Big] \\ &\cdot\exp\Big[\frac{i}{2}\int(\phi+J)D_c\,(\phi+J)\Big] \end{aligned} \tag{B.27}$$

where $J = g\,(J_1 + J_2)$. The quadratic ψ dependence in the exponential of (B.27), $\frac{i}{2}\int \partial\psi\cdot D_c\cdot\partial\psi = \frac{i}{2}\int\psi\Big[(-\partial^2)D_c\Big]\psi = \frac{i}{2}\int\psi^2$, exactly cancels the $\exp[-\frac{i}{2}\int\psi^2]$ factor of (B.25), so that integration over ψ then produces:

$$\frac{N_\theta^{-1}}{\det[n\cdot\partial D_c]}\,\delta\Big[\theta + (n\cdot\partial D_c)^{-1}(\partial D_c)(\phi+J)\Big] \tag{B.28}$$

The delta functional of (B.28) can now be used to evaluate the integrals over the remaining θ dependence of (B.27), so that the second line of (B.25) becomes:

$$\frac{N N_\psi N_A^{-1}}{\det[n\cdot\partial D_c]}\cdot\exp\Big[\frac{i}{2}\int(\phi+J)D^{ag}_c\,(\phi+J)\Big] \tag{B.29}$$

where $\left(\tilde{D}^{ag}_c\right)^{ab}_{\mu\nu} = \frac{\delta_{ab}}{k^2}\Big[\delta_{\mu\nu} - \frac{(n_\mu k_\nu + n_\nu k_\mu)}{k\cdot n} + \frac{n^2 k_\mu k_\nu}{(k\cdot n)^2}\Big]$ is the familiar, axial gauge propagator.

If one takes into account the fact that the functional integral:

$$\int d[\phi]\,\tilde{\mathcal{F}}[\phi]\,e^{\frac{i}{2}\int(\phi+J)D^{ag}_c\,(\phi+J)} \tag{B.30}$$

may be rewritten in terms of the linkage operator:

$$e^{\mathcal{D}_{ag}} \equiv \exp\Big[-\frac{i}{2}\int\frac{\delta}{\delta A}\,D_{ag}\,\frac{\delta}{\delta A}\Big]$$

as:

$$\begin{aligned} &e^{\mathcal{D}_{ag}}\,\mathcal{F}[A]\,e^{i\int A\cdot J}\Big|_{A\to 0} \\ &\quad = e^{\frac{i}{2}\int J\cdot D^{ag}_c\cdot J}\cdot e^{\mathcal{D}_{ag}}\,\mathcal{F}[A + \int D^{ag}_c J]\Big|_{A\to 0} \end{aligned} \tag{B.31}$$

our answer for (B.25) can be presented as:

$$\begin{aligned} e^{i\chi} = N'^2 &\int d[\alpha]\int d[u]\int d[\beta]\int d[v]\ e^{i\int[\alpha\cdot u+\beta\cdot v]} \\ &\cdot\left(e^{i\int\lambda^I\cdot u}\right)_+\left(e^{i\int\lambda^{II}\cdot v}\right)_+ \\ &\cdot e^{\frac{i}{2}\int J\,D^{ag}_c\,J}\cdot e^{\mathcal{D}_{ag}}\,\mathcal{F}[A+\int D^{ag}_c J]\Big|_{A\to 0} \\ &\cdot\left(e^{\mathcal{D}_{ag}}\,\mathcal{F}[A]\Big|_{A\to 0}\right)^{-1} \end{aligned} \tag{B.32}$$

upon using the overall normalization condition $\mathcal{Z}\{0,0,0\} = 1$, and the observation that the first line of (B.32) by itself – that is, without the (α,β) dependence in the J of (B.32)'s second line – is exactly unity. One sees from (B.32) that all disconnected graphs are automatically removed, with the eikonal function given by the sum of all connected, t channel gluonic exchanges between the scattering quarks. If one neglects interactions between exchanged gluons, as well as those of closed quark loops, $\mathcal{F}\to 1$, and we are in the realm of the simplest graphs of gluons exchanged between quarks, which we here use to define and illustrate the QAL.

The only other approximation needed is to replace the exponential factor of (B.32), $\frac{i}{2}\int J\, D_c^{ag}\, J$ by $ig^2 \int J_1\, D_c^{ag}\, J_2$, since we have already agreed to drop all self linkage terms. Then, since the currents $J_{1\mu}$ and $J_{2\nu}$ are proportional to $p_{1\mu}$ and $p_{2\nu}$, respectively, and since the QAL will subsequently replace the $\alpha_a(s)$ and $\beta_b(s)$ factors of J_1 and J_2 by $\alpha_a(0)$ and $\beta_b(0)$, the $\int ds_1$ and $\int ds_2$ of this exponential term will generate (using a Fourier representation of D_c) factors of $\delta(k \cdot p_1)$ and $\delta(k \cdot p_2)$, the only part of $(D_c^{ag})_{\mu\nu}^{ab}$ which remains is the "Feynman gauge" portion, $(\tilde{D}_c^{ag})_{\mu\nu}^{ab} \to (\tilde{D}_c)_{\mu\nu}^{ab} = \delta_{ab}\delta_{\mu\nu}/k^2$; and hence all the n_μ dependence disappears from this scattering amplitude, which is therefore independent of the gauge transformations available within the context of a general axial gauge. In this way, and for this simplest set of graphs, the QAL generates a result that is independent of gauge.

Finally, at this point we introduce a "gluon mass" into the propagator :

$$(\tilde{D}_c)_{\mu\nu}^{ab} \to (\tilde{\Delta}_c)_{\mu\nu}^{ab} = \delta_{ab}\, \delta_{\mu\nu}/(k^2 + \mu^2)$$

to represent in an *ad hoc* way any mass like structures, resulting from omitted radiative corrections to the virtual gluons exchanged between the scattering quarks. In a practical sense, we are suppressing the possibility of any IR singularities, a question that has already been extensively discussed in the first two papers of [6], and the references quoted there; and this is done because our eventual interest – although not realized in this paper – is to extract and identify any and all generalized, multiperipheral structure of the eikonal amplitude. In brief, we subsequently hope to learn just how close one can come to a "realistic Pomeron"; and, for this, IR singularities do not appear to be relevant.

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